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ON FIXED POINT THEOREM FOR 3-MAPPINGS IN FUZZY METRIC SPACES

Dr. U. K. Shrivastava[1], Neerja Namdeo[2]

[1] Atal Bihari Vajpayee Vishwavidyalaya, Bilaspur(C.G.), Govt. E.R.P.G.College, Bilaspur Distt - Bilaspur(C.G.), India.
[2] Pt. Ravishankar Shukla University, Raipur(C.G.), Govt. D. K. P. G. College, BalodaBazar Distt - BalodaBazar(C.G.), India.

ABSTRACT

The aim of this paper is to prove a common fixed point theorem on fuzzy metric space using the notion of occasionally weakly compatibility with assuming the completeness of the space. We extend the result of C.T.Aage and J.N.Salunke[1]

KEYWORDS: Fuzzy Metric Space, Occasionally weakly compatible mappings, Common fixed points.

INTRODUCTION

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh[10], which laid the foundation of fuzzy mathematics. In this paper, we deal with the fuzzy metric space defined by Kramosil and Michalek[5] and modified by George and Veeramani [2]. Sessa[7] introduced a generalization of commutativity, so called weak commutativity. Further Jungck[4] introduced more generalized commutativity which is compatibility in metric space and proved common fixed point theorems.

Recently, C.T. Aage and J.N. Salunke[1] introduced the concept of occasionally weakly compatible mappings in fuzzy metric space and proved common fixed point theorem. In this paper, we extend this concept for 3-mappings.Here we giving some definitions which are useful for our results.

Definition 1.1[8] A binary operation * : $[0,1]\times[0,1] \to [0,1]$ is continuous *t-norm* if * is satisfying the following conditions:

(i) * is commutative and associative,
(ii) * is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2[2] A 3- tuple (X, M, *) is called a *fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

For all $x, y, z \in X$ and $s, t > 0$

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \to (0,1]$ is continuous.

Then M is called a fuzzy metric on X. The function M(x, y, t) denote the degree of nearness between x and y with respect to t.

Example 1.3 Define $a * b = \min\{a, b\}$ and $M(x, y, t) = t/t- |x-y|$, for all $x, y \in X$ and $t>0$. Then (X, M, *) is a fuzzy metric space.

Definition 1.4[2] Let (X, M, *) be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X *converges* to x in X if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.
(b) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for each $p > 0$ and $t > 0$.
(c) A fuzzy metric space (X, M, *) is said to be *complete* if and only if every Cauchy sequence in X is convergent in X.

Definition 1.5[9] Two self mappings A and S of a fuzzy metric space (X, M, *) are called *compatible* if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$, for some x in X.

Definition 1.6[1] Let A and S are two self mappings of a fuzzy metric space (X, M, *). A point x in X is called a *coincidence point* of A and S if and only if Ax = Sx. We shall call w = Ax = Sx a point of coincidence of A and S.

Definition 1.7[4] Two self mappings A and S of a fuzzy metric space (X, M, *) are called *weakly compatible* if they commute at their coincidence points. i.e. if Au = Su for some $u \in X$, then ASu = SAu.

Definition 1.8[1] Two self mappings A and S of a fuzzy metric space (X, M, *) are called *occasionally weakly compatible (owc)* if and only if a point x in X which is coincidence point of A and S at which A and S commute.

Lemma 1.9[6] Let a fuzzy metric space (X, M, *). If for all $x, y \in X$ and $t > 0$ with positive number $q \in (0, 1)$ and $M(x, y, qt) \ge M(x, y, t)$, then $x = y$.

Lemma 1.10[1] Let A and S are two owc self mappings of a fuzzy metric space (X, M, *). If A and S have unique point of coincidence, w = Ax = Sx, then w is unique common fixed point of A and S.

Theorem A [1] Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs {A, S} and {B, T} be owc. If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \ge \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique point $w \in X$ such that Aw = Sw = w and a unique point $z \in X$ such that Bz = Tz = z. Moreover, z = w, so that there is a unique common fixed point of A, B, S and T.

Theorem B [1] Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs {A, S} and {B, T} be owc. If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \ge \Phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\})$$

for all $x, y \in X$ and $\Phi: [0,1] \to [0,1]$ such that $\Phi(t) > t$ for all $t \in (0, 1)$, then there exists a unique common fixed point of A, B, S and T.

Theorem C [1] Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs {A, S} and {B, T} be owc. If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \ge \Phi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t))$$

for all $x, y \in X$ and $\Phi: [0,1]5 \to [0,1]$ such that $\Phi(t, 1, 1, t, t) > t$ for all $t \in (0, 1)$, then there exists a unique common fixed point of A, B, S and T.

EXPERIENTIAL WORK

Theorem 2.1 Let (X, M, *) be a complete fuzzy metric space and let A, B, C, S, T and U be self mappings of X. Let the pairs {A, S}, {B, T} and {C, U} be occasionally weakly compatible (owc). If there exists $q \in (0, 1)$ such that

$$M(Ax, By, qt) \ge \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \quad ...(i)$$
$$M(By, Cz, qt) \ge \min\{M(Ty, Uz, t), M(Ty, By, t), M(Cz, Uz, t), M(By, Uz, t), M(Cz, Ty, t)\} \quad ...(ii)$$
$$M(Ax, Cz, qt) \ge \min\{M(Sx, Uz, t), M(Uz, Cz, t), M(Ax, Sx, t), M(Ax, Uz, t), M(Cz, Sx, t)\} \quad ...(iii)$$

for all x, y, z ϵ X and t > 0, then there exists a unique point u ϵ X such that Au = Su = u, a unique point v ϵ X such that Bv = Tv = v and a unique point w ϵ X such that Cw = Uw = w. Moreover, u = v = w, so that there is a unique common fixed point of A, B, C, S, T and U.

Proof: Let the pairs {A, S}, {B, T} and {C, U} be owc so there are three points x, y, z ϵ X such that Ax = Sx, By = Ty and Cz = Uz.

We have to show that
Part(I) (a) Ax = By
(b) By = Cz
And (c) Ax = Cz,

Part(II) (d) Uniqueness of u
(e) Uniqueness of v
And (f) Uniqueness of w,

Part(III) u = v = w.
Now we start the proof of our theorem.

Part(I)
(a) Let(hyp.) Ax ≠ By
By inequality (I)
M(Ax, By, qt) ≥ min{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)}
= min{M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), M(Ax, By, t), M(By, Ax, t)}
[since Ax = Sx, By = Ty]
= M(Ax, By, t)
Hence M(Ax, By, qt) ≥ M(Ax, By, t), for q ϵ (0, 1), t > 0.
Which give contradiction of lemma 1.9. Hence our hyp. is false and Ax = By.

(b) Let(hyp.) By ≠ Cz
By inequality (ii)
M(By, Cz, qt) ≥ min{M(Ty, Uz, t), M(Ty, By, t), M(Cz, Uz, t), M(By, Uz, t), M(Cz, Ty, t)}
= min{M(By, Cz, t), M(By, By, t), M(Cz, Cz, t), M(By, Cz, t), M(Cz, By, t)}
[since By = Ty, Cz = Uz]
= M(By, Cz, t)
Hence M(By, Cz, qt) ≥ M(By, Cz, t), for q ϵ (0, 1), t > 0.
Which give contradiction of lemma 1.9. Hence our hyp. is false and By = Cz.
Similarly, we can prove (c) Ax = Cz.

Part(II)
(a) Let(hyp.) u is not unique. Another point x1 ϵ X such that Ax1 = Sx1 then by(i) inequality, we have
Ax1 = Sx1 = By = Ty and Ax = By
Therefore Ax = Ax1 and u = Ax = Sx. Hence by lemma 1.10, u is the unique point of coincidence of A and S.
(b) Let(hyp.) v is not unique. Another point y1 ϵ X such that By1 = Ty1 then by(ii) inequality, we have
By1 = Ty1 = Cz = Uz and By = Cz
Therefore By = By1 and v = By = Ty. Hence by lemma 1.10, v is the unique point of coincidence of B and T.
Similarly, we can show (f), w is a unique point such that w = Cw = Uw.

Part(III)
Let(hyp.) u ≠ v, we have
M(u, v, qt) = M(Ax, By, qt)
≥ min{M(Su, Tv, t), M(Su, Au, t), M(Bv, Tv, t), M(Au, Tv, t), M(Bv, Su, t)}
= min{M(u, v, t), M(u, u, t), M(v, v, t), M(u, v, t), M(v, u, t)}
= M(u, v, t)

which gives contradiction of lemma 1.9.Therefore our hyp. is false and u =v. Similarly, we can show v = w. Hence by lemma 2.10, u = v = w and u is common fixed point of A, B, C, S, T and U. The uniqueness of the fixed point holds from (i), (ii), (iii).

Theorem 2.2 Let (X, M, *) be a complete fuzzy metric space and let A, B, C, S, T and U be self mappings of X. Let the pairs {A, S}, {B, T} and {C, U} be owc. If there exists q ϵ (0, 1) such that
M(Ax, By, qt) ≥ Φ(min{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)})
M(By, Cz, qt) ≥ Φ(min{M(Ty, Uz, t), M(Ty, By, t), M(Cz, Uz, t), M(By, Uz, t), M(Cz, Ty, t)})
M(Ax, Cz, qt) ≥ Φ(min{M(Sx, Uz, t), M(Uz, Cz, t), M(Ax, Sx, t), M(Ax, Uz, t), M(Cz, Sx, t)})
for all x, y, z ϵ X and Φ: [0,1] → [0,1] such that Φ(t) > t for all t ϵ (0, 1), then there exists a unique common fixed point of A, B, C, S, T and U.

Proof : The proof follows from Theorem 3.1.

Theorem 2.3 Let (X, M, *) be a complete fuzzy metric space and let A, B, C, S, T and U be self mappings of X. Let the pairs {A, S}, {B, T} and {C, U} be owc. If there exists q ϵ (0, 1) such that
M(Ax, By, qt) ≥ Φ(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t),M(Ax, Ty, t), M(By, Sx, t)) …(iv)
M(By, Cz, qt) ≥ Φ(M(Ty, Uz, t), M(Ty, By, t), M(Cz, Uz, t),M(By, Uz, t), M(Cz, Ty, t)) …(v)
M(Ax, Cz, qt) ≥ Φ(M(Sx, Uz, t), M(Uz, Cz, t), M(Ax, Sx, t), M(Ax, Uz, t), M(Cz, Sx, t)) …(vi)
for all x, y, z ϵ X and Φ: $[0,1]^5$ → [0,1] such that Φ(t, 1, 1, t, t) > t for all t ϵ (0, 1), then there exists a unique common fixed point of A, B, C, S, T and U.

Proof: Let the pairs {A, S}, {B, T} and {C, U} be owc so there are three points x, y, z ϵ X such that Ax = Sx, By = Ty and Cz = Uz.

We have to show that
Part(I) (a) Ax = By
(b) By = Cz
And (c) Ax = Cz.

Part(II) Uniqueness of point of coincidence u, v and w.
Now we start the proof of our theorem.

Part(I)
(a) Let(hyp.) Ax ≠ By
By inequality (iv)
M(Ax, By, qt) ≥ Φ(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t))
= Φ(M(Ax, By, t), M(Ax, Ax, t), M(By, By, t), M(Ax, By, t), M(By, Ax, t))
[since Ax = Sx, By = Ty]
= Φ(M(Ax, By, t), 1, 1, M(Ax, By, t), M(By, Ax, t))
> M(Ax, By, t) [since Φ(t, 1, 1, t, t) > t]
Hence M(Ax, By, qt) ≥ M(Ax, By, t), for q ϵ (0, 1), t > 0.
Which give contradiction of lemma 1.9. Hence our hyp. is false and Ax = By.

(b) Let(hyp.) By ≠ Cz
By inequality (v)
M(By, Cz, qt) ≥ Φ(M(Ty, Uz, t), M(Ty, By, t), M(Cz, Uz, t),M(By, Uz, t), M(Cz, Ty, t))
= Φ(M(By, Cz, t), M(By, By, t), M(Cz, Cz, t), M(By, Cz, t), M(Cz, By, t))
[since By = Ty, Cz = Uz]
= Φ(M(By, Cz, t), 1, 1, M(By, Cz, t), M(Cz, By, t))
> M(By, Cz, t) [since Φ(t, 1, 1, t, t) > t]
Hence M(By, Cz, qt) ≥ M(By, Cz, t), for q ϵ (0, 1), t > 0.
Which give contradiction of lemma 1.9. Hence our hyp. is false and By = Cz.

Similarly, we can prove (c) Ax = Cz.
We can easily show the proof of part (II) same as theorem 3.1, hence u = v = w and u is common fixed point of A, B, C, S, T and U. The uniqueness of the fixed point holds from (iv), (v), (vi).

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